Mathematical Notes

Elliptical Integrals

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Elliptical Integrals of I, II and III kinds, Ashgabat - Turkmenistan

(April 25, 2020)

Chapter 1

Elliptical Integrals of First Kind

$$\int \frac{dx}{\sqrt{(1-x)(1-k^2x^2)}} \tag{1.1}$$

$$\int \frac{dx}{\sqrt{(1-x)}\sqrt{(1-k^2x^2)}} = f(x), \qquad f'(x) = \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \tag{1.2}$$

Let‘s consider the form:

$$f'(h) = \frac{1}{(x(1-k^2x^2)h')^{1/2}} \tag{1.3}$$

$$\frac{1}{((1-h^2)\cancel{(1-k^2h^2)})^{1/2}} = \frac{1}{(x\cancel{(1-k^2h^2)}h')^{1/2}} \tag{1.4}$$

$$\frac{h'}{1-h^2} = \frac{1}{x} \qquad \frac{dh}{1-h^2} \quad \frac{1}{1-h^2} = \frac{1}{2}\left(\frac{1}{k-h} + \frac{1}{1+h}\right) \tag{1.5}$$

$$\frac{1}{2}(\ln(1+h) - \ln(1-h)) = \ln x \qquad \ln\left(\frac{1+h}{1-h}\right) = \ln x^2 \tag{1.6}$$

$$1+x^2 = \frac{-2}{h-2} \qquad h-1 = \frac{-2}{1+x^2} \qquad h = \frac{-2+1+x^2}{1+x^2} = \frac{x^2-1}{x^2+1} \tag{1.7}$$

$$h = 1 - \frac{2}{1+x^2} \tag{1.8}$$

$$f'(h) = \frac{1}{(x(1-k^2h^2)h)^{1/2}} \qquad h'f'(h) = \left(\frac{h'}{x(1-k^2h^2)}\right)^{1/2} \tag{1.9}$$

$$f(h) = \int \left(\frac{h'}{x(1-k^2h^2)}\right)^{1/2} dx \tag{1.10}$$

$$\frac{h'}{x} = \frac{2\cdot 2x}{\dfrac{(1+x^2)^2}{x}} = \frac{4}{(1+x^2)^2} \qquad \left(\frac{h'}{x}\right)^{1/2} = \frac{2}{1+x^2} \tag{1.11}$$

$$f(h) = \int \left(\frac{h'}{x}\right)^{1/2} \cdot \frac{1}{(1-k^2h^2)^{1/2}} = \int \frac{2}{1+x^2} \cdot \frac{1}{(1-k^2h^2)^{1/2}} \tag{1.12}$$

$$1 - \frac{k^2(x^2-1)^2}{(x^2+1)^2} = \frac{(x+1)^2 - k^2(x^2-1)^2}{(x^2+1)^2} \tag{1.13}$$

$$h = \frac{x^2-1}{x^2+1} f(h) = \int \frac{2}{\cancel{1+x^2}} \cdot \frac{\cancel{1+x^2}}{((x^2+1)-k^2(x^2-1))^{1/2}} \tag{1.14}$$

$$(x^2+1-k(x^2-1))(x^2+1+k(x^2-1))(x^2(1-k)+1+k)(x^2(1+k)+1-k) \tag{1.15}$$

$$\int \frac{2}{((x^2(v-k)+1+k)(x^2(1+k)+1-k))^{1/2}} \tag{1.16}$$

$$f(h) = \int \frac{2}{(1-k)^{1/2}\left(x^2+\frac{1+k}{1-k}\right)^{1/2}(1+k)^{1/2}\left(x^2+\frac{1-k}{1+k}\right)^{1/2}} \tag{1.17}$$

$$f(h) = \int \frac{2dx}{(1-k)^{1/2}\left(x^2+\frac{1+k}{1-k}\right)^{1/2}\left(x^2+\frac{1-k}{1+k}\right)^{1/2}} \tag{1.18}$$

Inteagral of type $\int \frac{dx}{(x^2+a^2)^{1/2}(x^2+b^2)^{1/2}}$ can be transformed to $\frac{1}{a}\int \frac{dt}{(1-t^2)^{1/2}\left(1-\frac{a^2-b^2}{a^2}t^2\right)^{1/2}}$ by substitution of: $x = \frac{bt}{\sqrt{1-t^2}}$, $a^2 = \frac{1+k}{1-k}$, $b^2 = \frac{1-k}{1+k}$, $b^2 = \frac{1}{a^2}$.

$$\frac{a^2-b^2}{a^2} = 1 - \frac{1}{a^4} = 1 - \frac{(1-k)^2}{(1+k)^2} = \frac{1+2k+k^2-1+2k-k^2}{(1+k)^2} = \frac{4k}{(1+k)^2} \tag{1.19}$$

$$f\left(h\left(\frac{bt}{\sqrt{1-t^2}}\right)\right) = \frac{2}{\cancel{(1-k)^{1/2}}+(1+k)^{1/2}} \cdot \frac{\cancel{(1-k)^{1/2}}}{(1+k)^{1/2}} \int \frac{dt}{(1-t^2)^{1/2}(1-(k')^2t^2)^{1/2}} \tag{1.20}$$

$$f(\varphi(t),k)=\frac{2}{1+k}\cdot f(t,k') \tag{1.21}$$

$$\varphi(t)=\frac{\dfrac{b^2t^2}{1-t^2}-1}{\dfrac{b^2t^2}{1+t^2}+1}=\frac{\dfrac{b^2t^2-1+t^2}{1-\cancel{t^2}}}{\dfrac{b^2t^2+1-t^2}{1-\cancel{t^2}}}=\frac{t^2(1+b^2)-1}{t^2(b^2-1)+1} \tag{1.22}$$

For special values $t=0,\quad \varphi(t)=-1,\quad t=1,\quad \varphi(t)=1$
From fundamental theorem of calculus:

$$\int_a^b=f(b)-f(a),f(1,k)-f(-1,k)=(f(1,k')-f(0,k'))\cdot\frac{2}{1+k} \tag{1.23}$$

$2K(k)=\dfrac{2}{1+k}K(k')$, since the integrand is even $K(k)=K(k')\cdot\dfrac{1}{1+k}$, Let's transform to form $f(t,k)=\dfrac{3}{1+k}f(g(t),k')$ where $\varphi og=t,\quad g(t)=\sqrt{\dfrac{1+t}{(1-b^2)t+b^2+1}}$

$$f(t,k)=\frac{2}{1+k}\cdot\frac{2}{1+k}\cdots\frac{2}{1+k_n}\cdot f(g_n(t),k_n) \tag{1.24}$$

where:
k_n - consecutive application of $\dfrac{2\sqrt{k}}{1+k}$ $g_n(t)-$ $\underbrace{\text{gogo...g(t)}}_{n\ times}$
Taking limit $lim_{n\to\infty}$:

$$lim_{n\to\infty}\frac{2}{1+k}\cdot\frac{2}{1+k_1}\cdots\frac{2}{1+k_n}=\frac{2}{\pi}K',\quad K'=K(\sqrt{1-k^2}) \tag{1.25}$$

$$lim_{n\to\infty}K_n=1,\quad lim_{n\to\infty}g_n(t)=u(t) \tag{1.26}$$

$$\begin{gathered}f(t,k)=\frac{2}{\pi}K'f(u(t),1)=\frac{2}{\pi}K'\int\frac{d(u(t))}{1-u^2(t)}=\frac{\cancel{2}K'}{\pi}\frac{1}{\cancel{2}}\ln\left(\frac{1+u(t)}{1-u(t)}\right)\\ f(t,k)=\frac{K'}{\pi}ln\left(\frac{1+u(t)}{1-u(t)}\right)\end{gathered} \tag{1.27}$$

Again:

$$\int_a^b = f(b,k) - f(a,k) \tag{1.28}$$

$$\begin{aligned}\int_0^t \frac{dt}{(1-t^2)^{1/2}(1-k^2t^2)^{1/2}} = f(t,k) - f(o,k) = \frac{K'}{\pi}\left(\left(\ln\frac{1+u(t)}{1-u(t)}\right) - \ln\left(\frac{1+u(o)}{1-u(o)}\right)\right) \qquad OR \\ = \frac{K'}{\pi}\left[\ln\left(\frac{1+g_n(t)}{1-g_n(t)}\right) - \ln\left(\frac{1+g_n(o)}{1-g_n(o)}\right)\right] \qquad where \quad n = \infty.\end{aligned} \tag{1.29}$$

Chapter 2

Elliptical Integrals of Second Kind

$$\int \sqrt{\frac{1-k^2x^2}{1-x^2}}dx \tag{2.1}$$

$$\int \frac{1-k^2x^2}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-k^2x^2}}dx = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} - k^2 \int \frac{x^2dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2.2}$$

$$\int \frac{x^2dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = f(x), \quad f'(x) = \frac{x^2}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2.3}$$

$$f'(h) = \frac{h^2}{(x(1-k^2h^2)h')^{1/2}} \tag{2.4}$$

$$\frac{h^2}{((1-h^2)(1-\cancel{k^2h^2}))^{1/2}} = \frac{h^2}{(x(1-\cancel{k^2h^2})h')^{1/2}} \tag{2.5}$$

$$\frac{h'}{1-h^2} = \frac{1}{x} \qquad h = 1 - \frac{2}{1+x^2} = \frac{x^2-1}{x^2+1} \tag{2.6}$$

$$f(h) = \int h^2 \left(\frac{h'}{x}\right)^{1/2} \cdot \frac{dx}{(1-k^2h^2)^{1/2}}, \qquad \left(\frac{h'}{x}\right)^{1/2} = \frac{2}{1+x^2} \tag{2.7}$$

$$1-k^2h^2=\frac{(1-k^2)\left(x^2+\dfrac{1+k}{1-k}\right)\left(x^2+\dfrac{1-k}{1+k}\right)}{(x^2+1)^2} \tag{2.8}$$

$$f(h)=\int\frac{(x^2-1)^2}{(x^2+1)^2}\cdot\frac{2}{\cancel{1+x^2}}\cdot\frac{(\cancel{x^2+1})dx}{(1-k^2)^{1/2}(x^2+a^2)^{1/2}(x^2+b^2)^{1/2}},\quad a^2=\frac{1+k}{1-k},\quad b^2=\frac{1-k}{1+k} \tag{2.9}$$

Again by substitution of $x=\dfrac{bt}{\sqrt{1-t^2}}$, we get:

$$f(h)=\frac{2}{a}\int\frac{h^2}{(1-k^2)^{1/2}}\cdot\frac{dt}{(1-t^2)^{1/2}(1-(k')^2t^2)^{1/2}} \tag{2.10}$$

where $k'=\dfrac{2\sqrt{k}}{1+k}$, $x=x(t)$

$$f(h)=\frac{2}{(1-k^2)^{1/2}}\cdot\frac{1}{a}\int\frac{h^2dt}{R'(t)},\quad R'(t)=(1-t^2)^{1/2}(1-(k')^2t^2)^{1/2} \tag{2.11}$$

Let's calculate $h^2(x(t))=\varphi^2(t)$

$$\varphi(t)=\frac{t^2(b^2+1)-1}{t^2(b^2-1)+1}=-\frac{1}{k}\left(1+B\cdot\frac{1}{t^3C}\right),\quad B=\frac{1-k^2}{2k},\quad C=\frac{1+k}{2k} \tag{2.12}$$

$$\varphi^2(t)=\frac{1}{k^2}\left(1+2B\frac{1}{t^2-C}+B^2\frac{1}{(t^2-C)^2}\right) \tag{2.13}$$

$$f(h)=\frac{2}{\cancel{(1-k)}^{1/2}(1+k)^{1/2}\dfrac{(1+k)^{1/2}}{\cancel{(1-k)^1/2}}}\int\frac{\varphi^2(t)dt}{R'} \tag{2.14}$$

$$f(h)=\frac{2}{1+k}\cdot\frac{1}{k^2}\int\left(\frac{1}{R'}+2B\cdot\frac{1}{t^3C}\cdot\frac{1}{R'}+B^2\cdot\frac{1}{(t^2-C)^2}\cdot\frac{1}{R'}\right)dt \tag{2.15}$$

Let's transform R':

$$(1-x^2)(1-(k')^2x^2)=1-(k')^2x^2-x^2+(k')^2x^4=1-((k')^2+1)x^2+(k')^2x^4 \tag{2.16}$$

If $R=\sqrt{\alpha+\beta+\gamma x^4}$

$$\frac{xR}{x^2+a} = \alpha_1 \int \frac{(x^2+a)dx}{R} + \beta_1 \int \frac{dx}{R} + \gamma_1 \int \frac{dx}{((x^2+a)R} + \delta_1 \int \frac{dx}{(x^2+a^2)R} \tag{2.17}$$

$$\alpha_1 = -(2n-5)\gamma, \quad \beta_1 = -(2n-4)(\beta - \delta a \gamma)$$

$$\gamma_1 = -(2n-3)(3a^2\gamma - 2a\beta + \alpha), \quad \delta_1 = (2n-2)(a^3\gamma - a^2\beta + a\alpha)$$

$$\alpha = 1 \qquad a = -C$$

$$\beta = -(1+(k')^2) \qquad n = 2 \qquad \gamma = (k')^2$$

$$1 \cdot (k') \int \frac{(x^2+a)dx}{R'} - 0 \cdot \int \frac{dx}{R'} - \underbrace{(3a^2(k')^2 + 2a(1+(k')^2) + 1)}_{A_1} \int \frac{dx}{(x^2+a)R'}$$

$$+2(\underbrace{a^3(k')^2 + a^2(1+(k')^2) + a \cdot 1}_{A_2}) \int \frac{dx}{(x^2+a)^2R'} = \frac{xR'}{x^2+a} \tag{2.18}$$

$$1+(k')^2 = \frac{1+4k}{(1+k)^2} = \frac{1+6k+k^2}{(1+k)^2}$$

$$A_1 = 3 \cdot \frac{\cancel{(1+k)^2}}{4k^2} \cdot \frac{4k}{\cancel{(1+K)^2}} - \frac{\cancel{2}\cancel{(1+k)}}{\cancel{2}k} \cdot \frac{1+6k+k^2}{(1+k)^2} + 1$$

$$= \frac{3}{k} - \frac{(1+6k+k^2)}{k(1+k)} + 1 = \frac{3+3k-1-6k-\cancel{k^2}+\cancel{k^2}+k}{k(1+k)} = \frac{2-2k}{k(1+k)} = \frac{2(1-k)}{k(1+k)}$$

$$A_1 = \frac{2(1-k}{k(1+k)} \tag{2.19}$$

$$A_2 = -\frac{(1+k)}{2k^2} + \frac{1+6k+k^2}{4k^2} - \frac{1+k}{2k} = \frac{-2-2k+1+6k+k^2-2k-2k^2}{4k^2} = \frac{-k^2+2k-1}{4k^2}$$

$$= \frac{-(k^2-2k+1)}{4k^2} = \frac{-(1-2)^2}{4k^2}$$

$$A_2 = \frac{-(1-k)^2}{4k^2} \tag{2.20}$$

From (2.15) we have:

$$f(h) = \frac{2}{1+k} \cdot \frac{1}{k^2} \int \frac{dx}{R'} + \frac{2}{1+k} \cdot \frac{1}{k^2} \cdot 2B \int \frac{dx}{(x^2+a)R'} + \frac{2B^2}{(1+k)} \cdot \frac{1}{k^2} \int \frac{dx}{(x^2+a^2)R'} \tag{2.21}$$

$$B^2 = \frac{(1-k)^2(1+k)^2}{4k^2} = -A_2(1+k)^2 \tag{2.22}$$

$$\frac{2B^2}{(1+k)k^2} = \frac{2}{\cancel{1+k}} \cdot \frac{1}{k^2} \cdot -A_2(1+k)^{\cancel{2}}$$

From (2.18) we have:

$$-2A_2 \int \frac{dx}{(x^2+a^2)R'} = -2A_2 \cdot \frac{1+k}{k^2} \int \frac{dx}{(x^2+a^2)R'} = \frac{1+k}{k^2} \cdot \left(-2A_2 \int \frac{dx}{(x^2+a^2)R'} \right) \tag{2.23}$$

So, from (**??**):

$$\begin{aligned} \frac{2B^2}{(1+k)k^2} \int \frac{dx}{(x^2+a^2)R'} &= -2A_2 \cdot \frac{1+k}{k^2} \int \frac{dx}{(x^2+a^2)R'} = \frac{1+k}{k^2} \cdot \left(-2A_2 \int \frac{dx}{(x^2+a^2)R'} \right) \\ &= \frac{4}{k(1+k)} \int \frac{x^2 dx}{R'} - \frac{2}{k^2} \int \frac{dx}{R'} - \frac{2(1-k)}{k^3} \cdot \int \frac{dx}{(x^2+a^2)R'} - \frac{xR'}{x^2+a} \cdot \frac{1+k}{k^2} \end{aligned} \tag{2.24}$$

Let's sum the corresponding integrals in (2.21): For $\int \frac{dx}{R'}$ the coefficient is: $\frac{2}{1+k} \cdot \frac{1}{k^2} = \frac{2}{k^2} \left(\frac{1}{1+k} - 1 \right) = \frac{2 \cdot \cancel{-k}}{\cancel{k} \cdot (1+k)} = \frac{-2}{k(1+k)}$

For $\int \frac{dx}{(x^2+a)R'} : \frac{4B}{(1+k)k^2} = \frac{\cancel{4}(1-2)(1+k)}{(1+k) \cdot k^2 \cdot \cancel{2}k} = \frac{2(1-k)}{k^3} - \frac{-2(1-k)}{k^3} = 0$

which simply vanishes.

So the expression (2.21) takes the form:

$$f(h) = \frac{-2}{k(1+k)} \int \frac{dx}{R'} + \frac{4}{k(1+k)} \int \frac{x^2 dx}{R'} - \frac{xR'}{x^2+a} \cdot \frac{1+k}{k^2} \tag{2.25}$$

$$h(x(t)) = \varphi(t) = \frac{t^2(b^2+1)-1}{t^2(b^2-1)+1} \tag{2.26}$$

$E(k)$ - for complete integral
For special values
$t = 0 \qquad \varphi = -1$
$t = 1 \qquad \varphi = 1$

1	1
-1	0

$$2E(k) = \frac{-2}{k(1+k)}k(k') + \frac{4}{k(1+k)}E(k') \qquad (2.27)$$

$$E(k) = \frac{-1}{k(1+k)}k(k') + \frac{2}{k(1+k)}E(k') \qquad (2.28)$$

Let's define:

$$\sigma(k) = K(k) - k^2E(k) = K(k) + \frac{k}{1+k}K(k') - \frac{-2}{1+k}E(k') = K(k') - \frac{2k}{1+k}E(k')$$
$$since \qquad K(k) = \frac{1}{1+k}K(k') \qquad (2.29)$$

$\sigma(k') = K(k') - (k')^2E(k')$

Let's multiply the above by $\frac{2}{1+k}$:

$$\frac{2}{1+k}\sigma(K) = \frac{2}{1+k}K(k') - (k')^2E(k') = \frac{2}{1+k}K(k') + \sigma(k') - K(k') \qquad (2.30)$$
$$since \qquad (k')^2 = \frac{4k}{(1+k)^2}$$
$$\frac{2}{1+k}\sigma(k) = \sigma(k') + (1-k)K(k) \qquad (2.31)$$

Define:

$$\int_0^x \frac{t^2dt}{R(t)} = E(x,k)$$
$$\int_0^x \frac{dt}{R(t)} = F(x,k), \qquad R(t) = (1-t^2)^{1/2}(1-k^2-t^2)^{1/2}$$

$$\sigma(x,k) = \int\left(\frac{1-k^2t^2}{1-t^2}\right)^{1/2}dt = \int_0^x \frac{dt}{R(t)} - k^2\int\frac{t^2dt}{R(t)} = F(x,k) - k^2E(x,k) \qquad (2.32)$$

$$h(x) = \varphi(x) = \frac{x^2(b^2+1)-1}{x^2(b^2-1)+1} \qquad \varphi(0) = -1 \qquad \varphi(1) = 1 \qquad (2.33)$$

$$f(\varphi(x),k) - f(-1,k) = \int_{-1}^{\varphi(x)} \frac{t^2dt}{R(t)} \qquad (2.34)$$
$$= \int_{-1}^{0}\frac{t^2dt}{R(t)} + \int_0^{\varphi(x)}\frac{t^2dt}{R(t)} = E(k) + E(\varphi(x),k)$$

$$E(k) + E(\varphi(x),k) = \frac{-2}{k(1+k)}\int_0^x \frac{dt}{R'(t)} + \frac{4}{k(1+k)}\int_0^x\frac{t^2dt}{R'(t)} - \frac{xR'}{x^2+a}\cdot\frac{1+k}{k^2} \qquad (2.35)$$

$$\sigma(h,k) = F(h,k) - k^2E(h,k) \tag{2.36}$$

$$= F(h,k) + \underbrace{\frac{2k}{1+k}F(x,k') - \frac{4k}{1+k}E(x,k')} + k^2E(k) + \frac{xR'}{x^2+a}\cdot(1+k)$$

$$\frac{2k}{1+k}F(x,k') + F(x,k')(1+k) - \frac{4k}{(1+k)^2}\cdot(1+k)E(x,k') - (1+k)F(x,k') \tag{2.37}$$

$$(1+k)(F(x,k') - (k')^2E(x,k')) \tag{2.38}$$

$$(1+k)\sigma(x,k') \tag{2.39}$$

$$\frac{2k}{1+k}\cdot F(x,k') - (1+k)\cdot F(x,k') = F(x,k')\left(\frac{2}{1+k)} - (1+k)\right) \tag{2.40}$$

$$= F(x,k')\left(\frac{\cancel{2k} - 1 - k^2 - \cancel{2k}}{1+k}\right)$$

$$= -F(x,k')\cdot\frac{1+k^2}{1+k}$$

$$*F(h,k) = \frac{2}{1+k}F(x,k') - K(k) \tag{2.41}$$

Let's sum the rest of equation:

$$\frac{2}{1+k}F(x,k') - K(k) - F(x,k')\cdot\frac{(1+k^2)}{1+k} + (1+k)\cdot\sigma(x,k') + k^2E(k) + \frac{xR'}{x^3+a}(1+k) \tag{2.42}$$

$$F(x,k')\left(\frac{2-1-k^2}{1+k}\right) = \frac{1-k^2}{1+k^2} + (1+k)\sigma(x,k') + k^2E(k) + \frac{xR'}{x^3+a}(1+k)$$

$$\sigma(h,k) = (1-k)(F(x,k') + (1+k)\sigma(x,k') - (K(k) - k^2E(k)) + \frac{xR'}{x^2+a}(1+k)$$

$$\sigma(h,k) = (1+k)\sigma(x,k') + (1-k)\cdot\sigma(x,k) - \sigma(k) + \frac{xR'}{x^2+a}(1+k)$$

$$where \qquad h(x) = \varphi(x) = \frac{x^2(b^2+1)-1}{x^2(b^2-1)+1}$$

$$R'(x) = ((1-x^2)(1-(k')^2x^2))^{1/2} \tag{2.43}$$

Chapter 3

Elliptical Integrals of Third Kind

$$\int \frac{dx}{(1-n^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}} \tag{3.1}$$

$$-\frac{1}{n^2}\sqrt{(1-x^2)(1-k^2x^2)}, \qquad m = -\frac{1}{n^2} < 0$$

$$\int \frac{dx}{(x^2+m)\sqrt{(1-x^2}(1-k^2x^2)} \tag{3.2}$$

$$\int \frac{dx}{(x^2+m)((1-x^2)^{1/2})(1-k^2x^2)^{1/2}} = f(x), \qquad f'(x) = \frac{1}{(x^2+m)(1-x^2)^{1/2}(1-k^2x^2)^{1/2}} \tag{3.3}$$

Let's consider the form:

$$f'(h) = \frac{1}{(h^2+m)[1-(k^2h^2)xh']^{1/2}}$$

(3.4)

$$f'(h) = \frac{1}{\cancel{(h^2+m)}} \cdot \frac{1}{(1-h^2)^{1/2}} \cdot \frac{1}{(1-k^2h^2)^{1/2}} = \frac{1}{\cancel{(h^2+m)}} \cdot \frac{1}{(1-k^2h^2)^{1/2}} \cdot \frac{1}{(xh')^{1/2}}$$

$$\frac{h'}{1-h^2} = x, \quad h = 1 - \frac{2}{1+x^2} = \frac{x^2-1}{x^2+1}$$

(3.5)

$$f(h) = \int \left(\frac{h'}{x}\right)^{1/2} \cdot \frac{1}{(h^3+m)} \cdot \frac{dx}{(1-k^2h^2)^{1/2}}$$

$$\left(\frac{h'}{x}\right)^{1/2} = \frac{2}{1+x^2}, \quad 1-k^2h^2 = 1 - \frac{k^2(x^2-1)^2}{(x^2+1)^2} = \frac{(x^2+1)^2 - k^2(x^2-1)^2}{(x^2+1)^2}$$

$$1-k^2h^2 = \frac{(x^2+1-k(x^2-1))(x^2+1+k(x^2-1))}{(x^2+1)^2} = \frac{x^2(1-k)+(1+k)(x^2(1+k)+(1-k))}{(x^2+1)^2}$$

$$= \frac{(1-k^2)\left(x^2+\frac{1+k}{1-k}\right)\left(x^2+\frac{1-k}{1+k}\right)}{(x^2+1)^2}$$

$$\frac{1}{h^2+m} = \frac{(x^2+1)^2}{m+1} \cdot \frac{1}{x^4+\alpha x^3+1}, \quad \alpha = \frac{2(m-1)}{m+1}$$

(3.6)

$$f(h) = \int \frac{2}{1+x^2} \cdot \frac{(x^2+1)^2}{m+1} \cdot \frac{1}{x^4+\alpha x+1}\frac{1}{(1-k^2)^{1/2}} \cdot \frac{(x^2+1)dx}{\left(x^2+\frac{1+k}{1-k}\right)^{1/2}\left(x^2+\frac{1-k}{1+k}\right)^{1/2}}$$

(3.7)

$$\frac{dx}{\left(x^2+\frac{1+k}{1-k}\right)^{1/2}\left(x^2+\frac{1-k}{1+k}\right)^{1/2}}$$ transforms by substitution:

$$x = \frac{bt}{\sqrt{1-t^2}}, \quad x^2 = \frac{b^2t^2}{1-t^2}, \quad a^2 = \frac{1+k}{1-k}, \quad b^2 = \frac{1-k}{1+k} \quad to$$

$$\frac{1}{a} \cdot \frac{dt}{\left(1-\frac{a^2-b^2}{a^2}t^2\right)^{1/2}(1-t^2)^{1/2}} = \frac{1}{a} \cdot \frac{1}{(1-t^2)^{1/2}} \cdot \frac{1}{(1-(k')t^2)^{1/2}}, \quad K' = \frac{2\sqrt{k}}{1+k}$$

$$f(h) = \int \frac{2}{m+1} \cdot \frac{(x^2+1)^2}{x^4+\alpha x^2+1} \cdot \frac{1}{(1-k^2)^{1/2}} \cdot \left(\frac{1-k}{1+k}\right)^{1/2} \cdot \frac{dt}{R'(t)} \tag{3.8}$$

$$R'(t) = (1-t^2)^{1/2}(1-(k')^2)t^2)^{1/2} \tag{3.9}$$

$$x(t) = \frac{bt}{\sqrt{1-t^2}} \tag{3.10}$$

$$f(h) = \frac{2}{(m+1)\cdot\frac{1}{1+k}} \int \left(\frac{1}{R'(t)} + \frac{(2-\alpha)x^2}{(x^4+\alpha x^2+1)} \cdot \frac{1}{R'(t)}\right) dt \tag{3.11}$$

Let's consider the second part of paranthesis: $x^2 = z$

$$\frac{z}{z^2+\alpha z+1} = \frac{A}{z-z_1} + \frac{B}{z-z_2}, \quad A = \frac{z_1}{z_1-z_2}, \quad B = \frac{z_2}{z_2-z_1}$$

$$z = \frac{-\alpha \pm \dfrac{4\sqrt{-m}}{m+1}}{2}$$

$x_1 = z_1, \quad x_2 = z_2$

$$z - z_1 = x^2 - x_1 = \frac{b^2t^2}{1-t^2} - x_1 = \frac{b^2t^2 - x_1 + x_1t^2}{1-t^2} = \frac{t^2(b^2+x_1)-x_1}{1-t^2}$$

$$(2-\alpha)\left(\frac{A}{z-z_1} + \frac{B}{z-z_2}\right) \cdot \frac{1}{R'(t)} \tag{3.12}$$

$$\begin{aligned}
\frac{A}{z-z_1} + \frac{B}{z-z_2} &= \frac{A(1-t^2)}{t^2(b^2+x_1)-x_1} + \frac{B(1-t^2)}{t^2(b^2+x_2)-x_2} \\
\frac{1-t^2}{t^2(b^2+x_1)-x_1} &= \frac{-1}{b^2+x_1} + \frac{b^2}{(b^2+x_1)^2} \cdot \frac{1}{t^2 - \dfrac{x_1}{b^2+x_1}} \\
\frac{1-t^2}{t^2(b^2+x_2)-X_2} &= \frac{-1}{b^2+x_2} + \frac{b^2}{(b^2+x_2)^2} \cdot \frac{1}{t^2 - \dfrac{x_2}{b^2+x_2}} \\
\frac{-x_1}{b^2+x-1} = \eta_1, &\quad \frac{-x_2}{b^2+x_2} = \eta_2
\end{aligned} \tag{3.13}$$

So, $\displaystyle\int \left(\frac{A}{z-z_1} + \frac{B}{z-z_2 \cdot \dfrac{1}{R'(t)}} \right) dt$ is equal to:

$$\frac{-A}{b^2+x_1}\int\frac{dt}{R'(t)} - B\cdot\frac{1}{b^2+x_2}\int\frac{dt}{R'(t)} + \frac{Ab^2}{(b^2+x_1)^2}\int\frac{1}{t^2+\eta_1}\cdot\frac{dt}{R'(t)} + \frac{Bb^2}{(b^2+x_2)^2}\int\frac{1}{t^2+\eta_2}\cdot\frac{dt}{R'(t)} \tag{3.14}$$

Let's evaluate the inside of paranthesis:

$$\begin{aligned}
\left(1-(1-\alpha)\left(\frac{A}{b62+x_1} + \frac{B}{b^2+x_2}\right)\right)\int\frac{dt}{R'(t)} \\
+\frac{(2-\alpha)\cdot Ab^2}{(b^2+x_2)^2}\int\frac{dt}{(t^2+\eta_1)(R'(t))} + \frac{(2-\alpha)Bb^2}{(b^2+x_2)^2}\int\frac{dt}{(t^2+\eta_2)(R'(t)} \\
C_1\int\frac{dt}{R'(t)} + C_2\int\frac{dt}{(t^2+\eta_1)R'(t)} + C_3\int\frac{dt}{(t^2+\eta_2)R'(t)}
\end{aligned} \tag{3.15}$$

C_1, C_2, C_3 - for correspnding coefficients let's change the notation:

$$p(t,m,k) = \int \frac{1}{t^2+m} \cdot \frac{dt}{R(t)} \tag{3.16}$$

$$f(t,k) = \int \frac{dt}{R(t)} \tag{3.17}$$

$$p(\varphi(t),m,k) = M(C_1 f(t,k') + C_2 p(t,n,k') + C_3 p(t,\eta_2,k')) \tag{3.18}$$

$$M = \frac{2}{(m+1)(1+k)} \tag{3.19}$$

Define:

$$\int_0^t \frac{dx}{R(x)} = F(t,k), \qquad \int_0^t \frac{1}{x^2+m} \cdot \frac{dx}{R(x)} = P(t,m,k) \tag{3.20}$$

$$\varphi(t) = \frac{t^2(b^2+1)-1}{t^2(b^2-1)+1}, \qquad so \quad \varphi(0) = -1 \tag{3.21}$$

$\varphi(t)$	t
-1	0

$$\int_{-} 1^{\varphi}(t)\frac{1}{t^2+m} + P(m,k) = MC_1F(t,k') + MC_2P(t,n,k') + MC_3P(t,\eta_2,k')$$

$$whereP(m,k)forcompleteingtegral$$

$$P(\varphi(t),m,k) = M_1C_1F(t,k') + MC_2P(t,n,k') + MC_3P(t,\eta_2,k') - P(m,k) \tag{3.22}$$

For special values $\varphi(t) = -1, \quad \varphi(t) = 1$

1	1
-1	0

$$2P(m,k) = M(C_1, K(k') + C_2P(\eta_1,k') + C_3P(\eta_2,k')) \tag{3.23}$$